M.Sc (Sem. 03) Unit – 03, Paper 11.

Reflexive spaces:— We know that every linear space L possesses a dual space L^* consisting of all linear functionals on L. Now L^* is also a linear space. Therefore it will possess a dual space $(L^*)^*$ consisting of all linear functionals on L^*. This dual space of L^* is called the second dual space of L and for the sake of simplicity we shall denote it by L^{**}.

Theorem:— Let $L(F)$ be a linear space. If x is any vector in L, the mapping f_x^* on L^* defined by

$$f_x^*(f) = f(x) \;\forall\; f \in L^*$$ is a linear functional on L^* i.e. $f_x^* \in L^*$.

Also the mapping $x \to f_x^*$ is an isomorphism of L into L^{**}.

Proof:— If $x \in L$ and $f \in L^*$, then $f(x)$ is a unique element of F. Therefore the correspondence f_x^* defined by

$$f_x^*(f) = f(x) \;\forall\; f \in L^* \quad \text{——— (1)}$$

is a mapping from L^* into F.

Let $\alpha, \beta \in F$ and $f, g \in L^*$. Then

$$f_x^*(\alpha f + \beta g) = (\alpha f + \beta g)(x) \quad \text{from (1)}.$$
$$= \alpha f(x) + \beta g(x)$$

$= \alpha\,[f_x^*(f)] + \beta\,[f_x^*(g)]$ from (1)

Therefore f_x^* is a linear functional on L^* and thus $f_x^* \in L^{**}$

Now let J be the mapping from L into L^{**} defined by $J(x) = f_x^* \ \forall\, x \in L$.

J is a Linear transformation.

Let $\alpha, \beta \in F$ and $x, y \in L$. Then

$J(\alpha x + \beta y) = f^*_{\alpha x + \beta y}$.

For every $f \in L^*$, we have

$$f^*_{\alpha x+\beta y}(f) = f(\alpha x + \beta y) \quad \text{(from (1))}$$

$$= \alpha f(x) + \beta f(y)$$

$$= \alpha f_x^*(f) + \beta f_y^*(f) \quad \text{(from (1))}$$

$$= (\alpha f_x^*)(f) + (\beta f_y^*)(f)$$

$$= (\alpha f_x^* + \beta f_y^*)(f)$$

$$\therefore f^*_{\alpha x+\beta y} = \alpha f_x^* + \beta f_y^* = \alpha J(x) + \beta J(y)$$

Thus $J(\alpha x + \beta y) = \alpha J(x) + \beta J(y)$. Therefore J is a linear transformation from L in L^{**}.

J is one-one. If $x, y \in L$, then $J(x) = J(y)$

$$\Rightarrow f_x^* = f_y^* \Rightarrow f_x^*(f) = f_y^*(f)\ \forall f \in L^*$$

$$\Rightarrow f(x) = f(y)\ \forall f \in L^*$$

$$\Rightarrow f(x) - f(y) = 0\ \forall f \in L^*$$

$$\Rightarrow f(x-y) = 0\ \forall f \in L^* \Rightarrow x - y = 0 \Rightarrow x = y$$

Therefore J is one-one.

Thus J is a linear transformation from L into L^{**} and J is one-one. Therefore J is an Isomorphism of L in L^{**}.

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